How to Analyze a 3D Curved Beam

Hao Yin, PhD Candidate

Department of Civil and Environmental Engineering

Sep 17, 2019

Outline

Classical Beam Theories

- In classical beam theories, beam is essentially an one-dimensional object (only the beam axis exists).
- Assumptions made in both Euler-Bernoulli beam theory and Timoshenko beam theory:
    1. Plane sections remain plane during deformation (rigid cross sections).
    2. Plane sections normal to the beam axis in the original configuration.

Classical Beam Theories

- Euler-Bernoulli beam v.s. Timoshenko beam

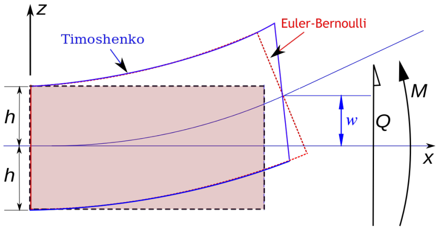


Figure 1: Deformation of a Timoshenko beam (blue) compared with that of an Euler-Bernoulli beam (red)

- If we assume that the plane sections always normal to the beam axis during the deformation...
- If the shear deformation is taken into account...

Classical Beam Theories

- Curved Timoshenko beam
  - How to describe an arbitrary curve in space?
  - Parametric curve

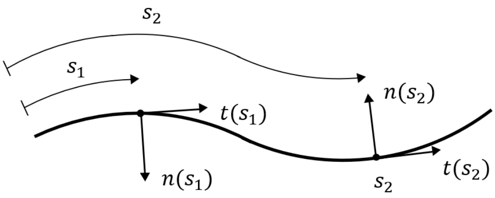


Figure 2: Parametric curve and local system of references in 2D

Classical Beam Theories

- Parametric curve
  - Express the coordinates of the points of the curve as functions of variables, called parameters

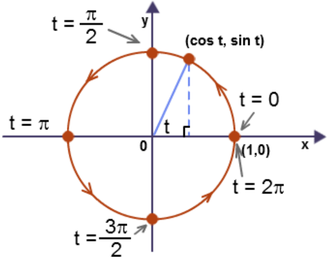


Figure 3: Parametric curve and location system of references in 2D

- One simple example of parametric curve is a circle in Cartesian coordinate system, two parameters are radius r and angle t

$$\begin{aligned} x &= r\cos t \\ y &= r\sin t \end{aligned} \tag{1}$$

Classical Beam Theories

- Curved Timoshenko beam
  - How to describe an arbitrary curve in space?
  - Parametric curve - parameterized by the arc-length s

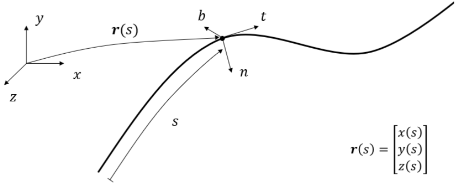


Figure 4: Parametric curve and location system of references in 3D

- 3D cases: Frenet-Serret (TNB) frame
  - Introduced to describe the kinematic properties of a point moving along the 3D curve, or the geometric properties of the curve itself

$$\begin{bmatrix} \frac{d\boldsymbol{t}}{ds} \\ \frac{d\boldsymbol{n}}{ds} \\ \frac{d\boldsymbol{b}}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{2}$$

Classical Beam Theories

- $\boldsymbol{t}$ defines the direction of the cross section (plane normal), $\boldsymbol{n}$ and $\boldsymbol{b}$ define the local direction in plane.
- Now with the help of the only parameter - arc length s, and the information of local directions from Frenet-Serret frame, we can express everything as functions of s.
- For example: position $\boldsymbol{r}$(s), displacement $\boldsymbol{u}$(s) and force $\boldsymbol{Q}$(s) etc.

$$\boldsymbol{r}(s) = \begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix}, \boldsymbol{u}(s) = \begin{bmatrix} u_t(s) \\ u_n(s) \\ u_b(s) \end{bmatrix}, \boldsymbol{Q}(s) = \begin{bmatrix} N(s) \\ Q_n(s) \\ Q_b(s) \end{bmatrix} \qquad (3)$$

Classical Beam Theories

- 3D curved Timoshenko beam problem

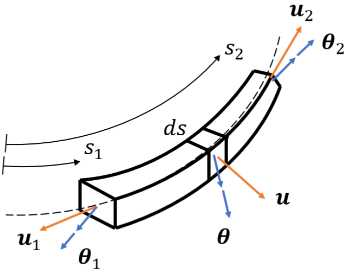


Figure 5: Displacements and rotations

- Kinematics

$$\begin{aligned}\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1 &= \int_{s_1}^{s_2} \boldsymbol{\chi}(s)\mathrm{d}s \\ \boldsymbol{u}_2 - \boldsymbol{u}_1 - \int_{s_1}^{s_2} (\boldsymbol{\theta} \times \boldsymbol{t})ds &= \int_{s_1}^{s_2} \boldsymbol{\varepsilon}(s)ds\end{aligned} \tag{4}$$

Classical Beam Theories

- 3D curved Timoshenko beam problem

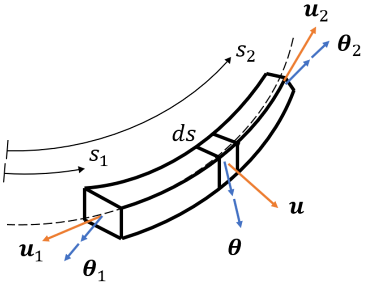


Figure 5: Displacements and rotations

- Kinematics

$$\varepsilon(s) = \frac{d\boldsymbol{u}}{ds} - \boldsymbol{\theta} \times \boldsymbol{t}$$
$$\chi(s) = \frac{d\boldsymbol{\theta}}{ds} \tag{5}$$

Classical Beam Theories

- 3D curved Timoshenko beam problem

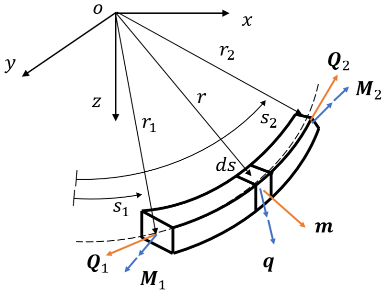


Figure 6: Internal and applied forces

- Equilibrium

$$\begin{aligned} \boldsymbol{Q}_2 - \boldsymbol{Q}_1 + \int_{s_1}^{s_2} \boldsymbol{q}(s)\mathrm{d}s = \mathbf{0} \\ \boldsymbol{M}_2 - \boldsymbol{M}_1 + (\boldsymbol{r}_2 \times \boldsymbol{Q}_2) - (\boldsymbol{r}_1 \times \boldsymbol{Q}_1) + \int_{s_1}^{s_2} (\boldsymbol{r} \times \boldsymbol{q} + \boldsymbol{m}) ds = \mathbf{0} \end{aligned} \tag{6}$$

Classical Beam Theories

- 3D curved Timoshenko beam problem

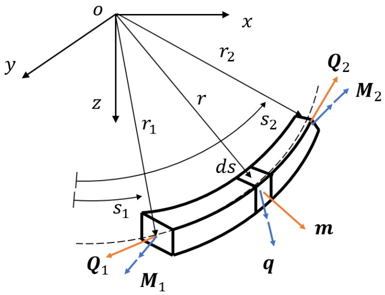


Figure 6: Internal and applied forces

- Equilibrium

$$
\begin{aligned}
\frac{d\boldsymbol{Q}}{ds} + \boldsymbol{q} &= \boldsymbol{0} \\
\frac{d\boldsymbol{M}}{ds} + \boldsymbol{t} \times \boldsymbol{Q} + \boldsymbol{m} &= \boldsymbol{0}
\end{aligned}
\tag{7}
$$

Classical Beam Theories

- Constitutive law (Linear, isotropic, elastic)

$$\begin{bmatrix} Q_t \\ Q_n \\ Q_b \\ M_t \\ M_n \\ M_b \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GA_n & 0 & 0 & 0 & 0 \\ 0 & 0 & GA_b & 0 & 0 & 0 \\ 0 & 0 & 0 & GI_t & 0 & 0 \\ 0 & 0 & 0 & 0 & EI_n & 0 \\ 0 & 0 & 0 & 0 & 0 & EI_b \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \varepsilon_n \\ \varepsilon_b \\ \chi_t \\ \chi_n \\ \chi_b \end{bmatrix} \tag{8}$$

- Combining equation (5)(7)(8) gives the governing equations of a 3D curved Timoshenko beam.

Classical Beam Theories

- But the classical beam theories are derived based on the condition of the whole cross section.
- Can we find more if we analyze the points which are off the beam axis themselves?

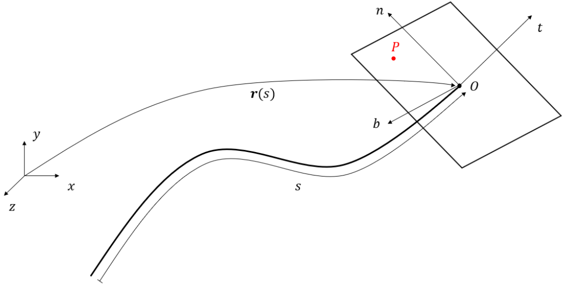


Figure 7: A generic point P locating off the center line of the beam

New Beam formulation

- Position vector p

$$\begin{aligned} \boldsymbol{x}(s, p_t, p_n, p_b) &= \boldsymbol{r}(s) + \boldsymbol{p} \\ &= \boldsymbol{r}(s) + p_t \boldsymbol{t} + p_n \boldsymbol{n} + p_b \boldsymbol{b} \end{aligned} \tag{9}$$

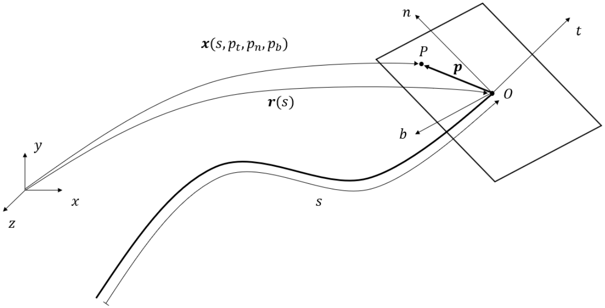


Figure 8: A generic point P locating off the center line of the beam

New Beam formulation

- According to the assumption made in beam theories that the cross sections normal to the beam axis before deformation

$$\begin{aligned} \boldsymbol{p} \cdot \boldsymbol{t} &= 0 \\ p_t &= 0 \end{aligned} \tag{10}$$

- $[p_t, p_n, p_b] \rightarrow [0, p_n, p_b]$

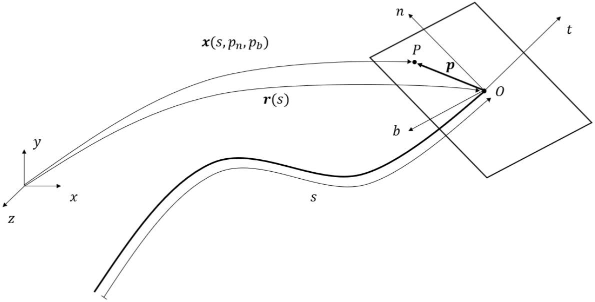


Figure 9: A generic point P locating off the center line of the beam (continued)

New Beam formulation: Kinematics

- the total displacement of point P can be divided into two parts:
  - displacement due to rigid body translation of the cross section $\boldsymbol{\Delta u}$
  - displacement due to rigid body rotation of the cross section $\boldsymbol{\Delta p}$

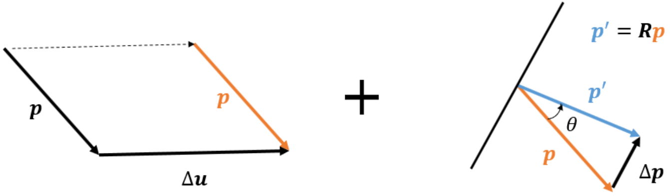


Figure 10: displacement decomposition

- displacement due to rigid body translation of the cross section can be represented by the displacement at the center of the cross section

$$\boldsymbol{\Delta u} = \boldsymbol{u}_0 \tag{11}$$

New Beam formulation: Kinematics

- displacement due to rigid body rotation of the cross section $\boldsymbol{\Delta p}$, according to Rodrigues' rotation formula $\boldsymbol{p'} = \boldsymbol{Rp}$

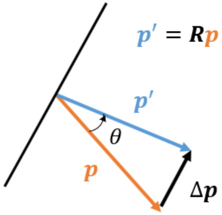


Figure 11: Rodrigues' rotation formula

$$
\begin{aligned}
\boldsymbol{\Delta p} &= \boldsymbol{p'} - \boldsymbol{p} \\
&= (\boldsymbol{R} - \boldsymbol{I})\boldsymbol{p} \\
&= [(\sin\theta)\boldsymbol{K} + (1-\cos\theta)\boldsymbol{K}^2]\boldsymbol{p} \\
&\approx \theta\boldsymbol{K}\boldsymbol{p} = \boldsymbol{\theta} \times \boldsymbol{p}
\end{aligned}
\tag{12}
$$

New Beam formulation: Kinematics

hence the total displacement u at the material point P

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{\theta} \times \boldsymbol{p} \tag{13}$$

New Beam formulation: Compatibility

- Recall in Cartesian coordinate system, strain tensor

$$\varepsilon = \frac{1}{2}(\nabla_{\boldsymbol{X}} \boldsymbol{u} + \nabla_{\boldsymbol{X}} \boldsymbol{u}^T) \tag{14}$$

- But our coordinate system is not a Cartesian coord system anymore - a curvilinear coordinate system instead.

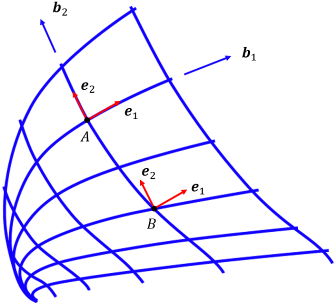


Figure 12: Change in directions of local basis in curvilinear coordinate system

New Beam formulation: Compatibility

- Hence equation (14) can not be applied directly.
- Luckily, local mapping from curvilinear coordinate system to Cartesian coordinate system at generic point exists, here we apply the inverse mapping of gradients

$$\begin{aligned} \nabla_{\boldsymbol{t}} \boldsymbol{u} &= \nabla_{\boldsymbol{X}} \boldsymbol{u} \cdot \boldsymbol{J} \\ \nabla_{\boldsymbol{X}} \boldsymbol{u} &= \nabla_{\boldsymbol{t}} \boldsymbol{u} \cdot \boldsymbol{J}^{-1} \end{aligned} \tag{15}$$

- Calculate $\nabla_{\boldsymbol{t}} \boldsymbol{u}$ and $\boldsymbol{J}$ and substitute all back to strain tensor

$$\begin{aligned} \boldsymbol{\varepsilon} &= \frac{1}{J}(\boldsymbol{\varepsilon}_0 + \boldsymbol{\chi} \times \boldsymbol{p}) \\ \boldsymbol{\chi} &= \frac{d\boldsymbol{\theta}}{ds} \end{aligned} \tag{16}$$

New Beam formulation: Equilibrium

- Following the principal of virtual work, the variation of internal work

$$
\begin{aligned}
\delta W_{int} &= \int_V \left(\sigma_{tt}\delta\varepsilon_{tt} + \tau_{tn}\delta\gamma_{tn} + \tau_{tb}\delta\gamma_{tb}\right) dV \\
&= \int_V \left(\sigma_{tt}\delta\varepsilon_{tt} + \tau_{tn}\delta\gamma_{tn} + \tau_{tb}\delta\gamma_{tb}\right) J dp_t dp_n dp_b \\
&= \int_l \int_A \left(\sigma_{tt}\delta\varepsilon_{tt} + \tau_{tn}\delta\gamma_{tn} + \tau_{tb}\delta\gamma_{tb}\right) J dA ds
\end{aligned}
\tag{17}
$$

the variation of external work

$$
\delta W_{ext} = \int_l \left(q_t \delta u_{0t} + q_n \delta u_{0n} + q_b \delta u_{0b} + m_t \delta\theta_t + m_n \delta\theta_n + m_b \delta\theta_b\right) ds
\tag{18}
$$

New Beam formulation: Equilibrium

- recall the definition of stress resultants (e.g. $N = \int_A \sigma_{tt} dA$, $M_n = \int_A \sigma_{tt} p_b dA$), then we can derive the equilibrium

$$
\begin{aligned}
\left(\frac{dN}{dp_t} - \kappa Q_n\right) + q_t = 0 \\
\left(\kappa N + \frac{dQ_n}{dp_t} - \tau Q_b\right) + q_n = 0 \\
\left(\tau Q_N + \frac{dQ_b}{dp_t}\right) + q_b = 0 \\
\left(\frac{dM_t}{dp_t} - \kappa M_n\right) + m_t = 0 \\
\left(\kappa M_t + \frac{dM_n}{dp_t} - \tau M_b\right) - Q_b + m_n = 0 \\
\left(\tau M_n + \frac{dM_b}{dp_t}\right) + Q_n + m_b = 0
\end{aligned} \tag{19}
$$

New Beam formulation: Governing equations

- If linear elastic...

$$N = \int_A \sigma_{tt} dA = E \int_A \varepsilon_{tt} dA$$
$$= E\left(\frac{du_{0t}}{dp_t} - \kappa u_{0n}\right) \int_A \frac{1}{1-\kappa p_n} dA + ... \quad (20)$$

- define the equivalent cross sectional properties, for example

$$A^* = \int_A \frac{1}{1-\kappa p_n} dA \quad (21)$$

New Beam formulation: Governing equations

$$
\begin{aligned}
&\frac{d}{ds}\left[EA^*\left(\frac{du_{0t}}{ds}-\kappa u_{0n}\right)+ES_n^*\left(\kappa\theta_t+\frac{d\theta_n}{ds}-\tau\theta_b\right)-ES_b^*\left(\tau\theta_n+\frac{d\theta_b}{ds}\right)\right]\\
&\quad-\kappa\left[GA_n^*\left(\kappa u_{0t}+\frac{du_{0n}}{ds}-\tau u_{0b}-\theta_b\right)-GS_n^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]+q_t=0\\
&\kappa\left[EA^*\left(\frac{du_{0t}}{ds}-\kappa u_{0n}\right)+ES_n^*\left(\kappa\theta_t+\frac{d\theta_n}{ds}-\tau\theta_b\right)-ES_b^*\left(\tau\theta_n+\frac{d\theta_b}{ds}\right)\right]\\
&\quad+\frac{d}{ds}\left[GA_n^*\left(\kappa u_{0t}+\frac{du_{0n}}{ds}-\tau u_{0b}-\theta_b\right)-GS_n^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]\\
&\quad-\tau\left[GA_b^*\left(\tau u_{0n}+\frac{du_{0b}}{ds}+\theta_n\right)+GS_b^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]+q_n=0\\
&\tau\left[GA_n^*\left(\kappa u_{0t}+\frac{du_{0n}}{ds}-\tau u_{0b}-\theta_b\right)-GS_n^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]\\
&\quad+\frac{d}{ds}\left[GA_b^*\left(\tau u_{0n}+\frac{du_{0b}}{ds}+\theta_n\right)+GS_b^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]+q_b=0\\
&\frac{d}{ds}\left[GS_b^*\left(\tau u_{0n}+\frac{du_{0b}}{ds}+\theta_n\right)-GS_n^*\left(\kappa u_{0t}+\frac{du_{0n}}{ds}-\tau u_{0b}-\theta_b\right)+GI_t^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]\\
&\quad-\kappa\left[ES_n^*\left(\frac{du_{0t}}{ds}-\kappa u_{0n}\right)+EI_{nn}^*\left(\kappa\theta_t+\frac{d\theta_n}{ds}-\tau\theta_b\right)-EI_{nb}^*\left(\tau\theta_n+\frac{d\theta_b}{ds}\right)\right]+m_t=0\\
&\kappa\left[GS_b^*\left(\tau u_{0n}+\frac{du_{0b}}{ds}+\theta_n\right)-GS_n^*\left(\kappa u_{0t}+\frac{du_{0n}}{ds}-\tau u_{0b}-\theta_b\right)+GI_t^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]\\
&\quad+\frac{d}{ds}\left[ES_n^*\left(\frac{du_{0t}}{ds}-\kappa u_{0n}\right)+EI_{nn}^*\left(\kappa\theta_t+\frac{d\theta_n}{ds}-\tau\theta_b\right)-EI_{nb}^*\left(\tau\theta_n+\frac{d\theta_b}{ds}\right)\right]\\
&\quad+\tau\left[ES_b^*\left(\frac{du_{0t}}{ds}-\kappa u_{0n}\right)+EI_{nb}^*\left(\kappa\theta_t+\frac{d\theta_n}{ds}-\tau\theta_b\right)-EI_{bb}^*\left(\tau\theta_n+\frac{d\theta_b}{ds}\right)\right]\\
&\quad-\left[GA_b^*\left(\tau u_{0n}+\frac{du_{0b}}{ds}+\theta_n\right)+GS_b^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]+m_n=0\\
&\tau\left[ES_n^*\left(\frac{du_{0t}}{ds}-\kappa u_{0n}\right)+EI_{nn}^*\left(\kappa\theta_t+\frac{d\theta_n}{ds}-\tau\theta_b\right)-EI_{nb}^*\left(\tau\theta_n+\frac{d\theta_b}{ds}\right)\right]\\
&\quad-\frac{d}{ds}\left[ES_b^*\left(\frac{du_{0t}}{ds}-\kappa u_{0n}\right)+EI_{nb}^*\left(\kappa\theta_t+\frac{d\theta_n}{ds}-\tau\theta_b\right)-EI_{bb}^*\left(\tau\theta_n+\frac{d\theta_b}{ds}\right)\right]\\
&\quad+\left[GA_n^*\left(\kappa u_{0t}+\frac{du_{0n}}{ds}-\tau u_{0b}-\theta_b\right)-GS_n^*\left(\frac{d\theta_t}{ds}-\kappa\theta_n\right)\right]+m_b=0
\end{aligned}
$$

Implementation with Isogeometric Analysis

- Isogeometric Analysis (IGA) can be seen as an extension of Finite Element Method (FEM).
- Non-Uniform Rational B-Splines (NURBS) are used in IGA as basis functions for Finite Element, which are commonly used for the geometry description in CAD.
- More convenient, more accurate version of FEM.

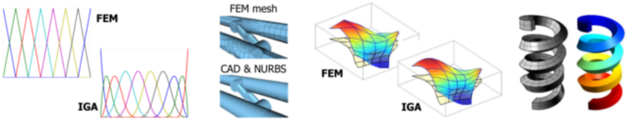


Figure 13: Isogeometric Analysis v.s. Finite Element Method

Implementation with Isogeometric Analysis

- Since our new beam formulation is a "quasi-3D" formulation, to verify this new formulation, we compared the results in IGA with beam elements with the results in IGA with solid elements.

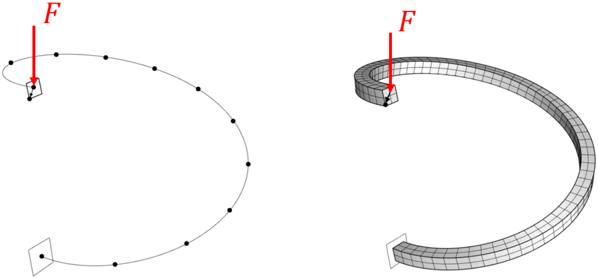


Figure 14: Spiral staircase- beam elements (left), solid elements (right)

Implementation with Isogeometric Analysis

- The new beam formulation is directly derived from the compatibility conditions at generic points, for thick beam, it performs better than classical beam formulations.

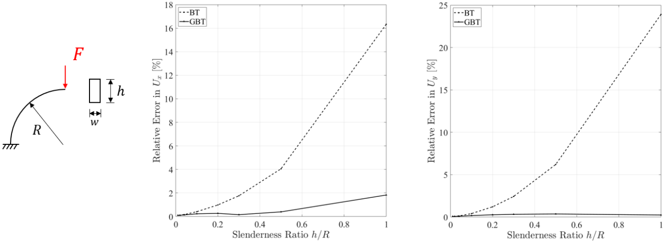


Figure 15: quarter circle beam (left), relative error in U_x (mid), relative error in U_y (right)

Implementation with Isogeometric Analysis

- The position of the beam axis now can be picked arbitrarily.

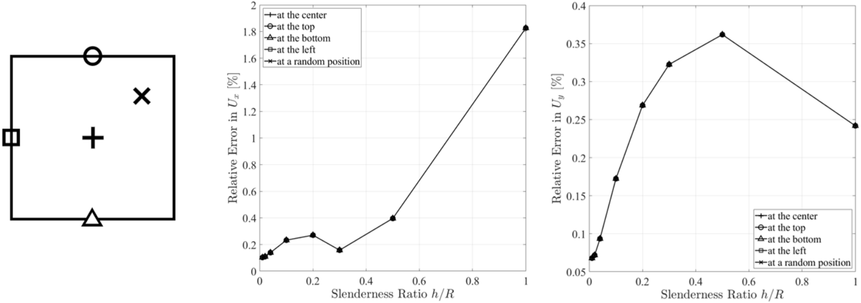


Figure 16: Locations of the beam axis (left), relative error in U_x (mid), relative error in U_y (right)

Implementation with Isogeometric Analysis

- Which makes the following shape of cross section possible.

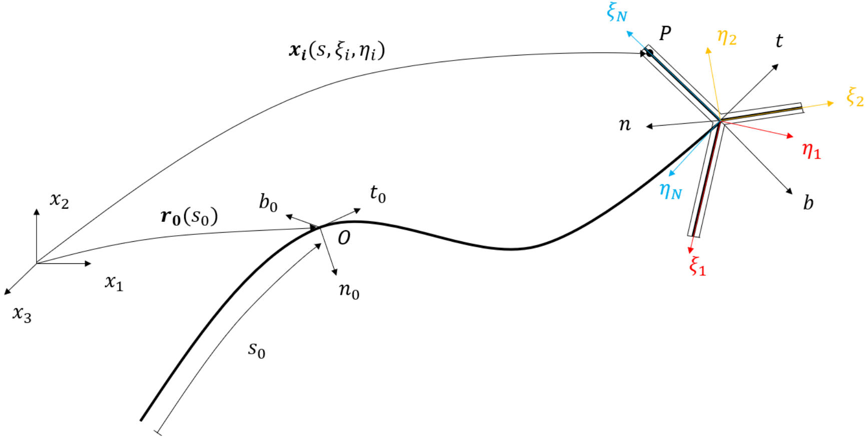
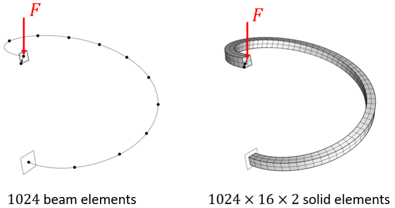


Figure 17: Typical shape of curved beams in the wood lattice model

Implementation with Isogeometric Analysis

- For our wood lattice model, since we need to model many thousands - even a million beams simultaneously, it has to be computationally efficient.



Number of elements in longitudinal direction	beam	solid
16	0.87	3.55
64	0.97	16.73
256	2.56	106.89
1024	8.99	32324.14

Figure 18: Running time (unit: second)

Implementation with Isogeometric Analysis

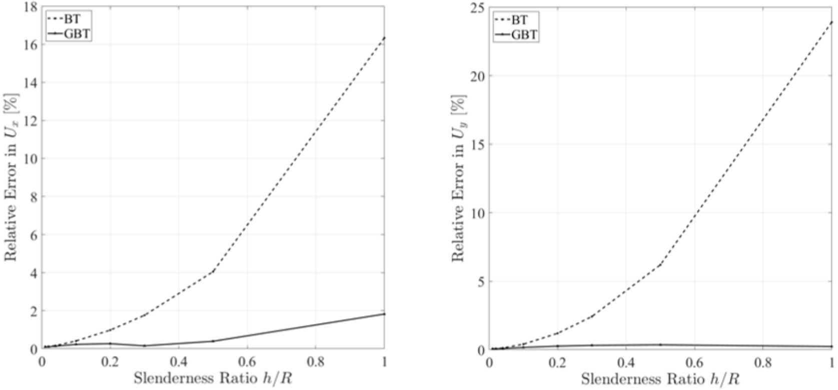


Figure 19: Convergence study

Questions?

Hao Yin
haoyin2022@u.northwestern.edu
www.cusatis.us